

ON UNIQUENESS OF CHARACTERISTIC CLASSES

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ABSTRACT. We give an axiomatic characterization of maps from algebraic K-theory. The results apply to a class of maps from algebraic K-theory to any suitable cohomology theory or to algebraic K-theory, which includes all group morphisms. In particular, we obtain comparison theorems for the Chern character and Chern classes and for the Lambda and Adams operations on higher algebraic K-theory. We show that the Adams operations defined by Grayson agree with the ones defined by Gillet and Soulé.

INTRODUCTION

In this paper we address the problem of comparing maps from the algebraic K -groups of a scheme to either algebraic K -groups or suitable cohomology theories. This type of questions often arise when one constructs a map that is supposed to induce, in some particular cohomology theory, a specific regulator or known map, and one needs to show that the map is indeed the expected one. Examples of these situations are found in the construction of the Beilinson regulator given by Burgos and Wang in [BW98] or in the definition of the Adams operations given by Grayson in [Gra92].

The different nature of each construction usually makes direct comparisons not an available option and one is forced to turn to theoretical tricks. In this work, we identify sufficient conditions for two maps to agree, thus obtaining an axiomatic characterization of maps from K -theory.

In abstract, the results apply to a class of maps, named *weakly additive*. All group morphisms induced by a map of sheaves are in this class, but are not the only ones. As a main consequence, we give a characterization of the *Adams* and *lambda operations* on higher K -theory and of the *Chern character* and *Chern classes* on a suitable cohomology theory (see section 5.2).

In particular, we show that the Adams operations defined by Grayson in [Gra92] agree with the ones defined by Gillet and Soulé in [GS99], for all noetherian schemes of finite Krull dimension. This implies that for this class of schemes, the operations defined by Grayson satisfy the usual identities of a lambda ring. This is an original result.

The second specific application of this work is a proof that the regulator defined by Burgos and Wang in [BW98] is the Beilinson regulator. The proof provided here is simpler than the one given in loc. cit., where delooping in K -theory was required.

The results of this paper are further exploited in the paper under preparation by the author ([Fel08],[Fel07]), where an explicit chain morphism representing the Adams operations on higher algebraic K -theory with rational coefficients is constructed. Furthermore, in [Fel07], [BF08], Burgos and the author defined a morphism in the derived category of complexes from a chain complex computing higher algebraic Chow groups to Deligne-Beilinson cohomology. A slight modification of the tools developed here allows one to prove that this morphism induces the Beilinson regulator. This result ultimately implies that the regulator defined by Goncharov in

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[Gon05] induces the Beilinson regulator as well (work under preparation by Burgos, Feliu and Takeda).

The techniques used in this paper rely on the generalized cohomology theory described by Gillet and Soulé in [GS99]. Roughly speaking, the idea is that any good enough map from K -theory to K -theory or to a cohomology theory is characterized by its behavior over the K -groups of the simplicial classifying scheme $B.GL_N$.

More explicitly, let \mathbf{C} be the big Zariski site over a noetherian finite dimensional scheme S . For any integers $N, k \geq 0$ denote by $B.GL_{N/S}$ the simplicial scheme $B.GL_N \times_{\mathbb{Z}} S$ and let $Gr_S(N, k)$ be the Grassmanian scheme over S . If $S.\mathcal{P}$ denotes the Waldhausen simplicial sheaf computing algebraic K -theory, for every scheme X in \mathbf{C} and for all $m \geq 0$, we have $K_m(X) \cong \pi_{m+1}(S.\mathcal{P}(X))$. The definition of algebraic K -groups can be extended to simplicial schemes over \mathbf{C} and every sheaf map (in the homotopy category of simplicial sheaves) $S.\mathcal{P} \rightarrow S.\mathcal{P}$ induces a map $K_m(Y) \rightarrow K_m(Y)$ for every simplicial scheme Y .

Let \mathbb{F} be a simplicial sheaf over \mathbf{C} . The two main consequences of our uniqueness theorem are the following.

- (i) Assume that \mathbb{F} is weakly equivalent to $S.\mathcal{P}$, and let $\Phi, \Phi' : S.\mathcal{P} \rightarrow \mathbb{F}$ be two H -space maps. Then, the morphisms

$$\Phi, \Phi' : K_m(X) \rightarrow K_m(X), \quad m \geq 0$$

agree for all schemes X in \mathbf{C} , if they agree over $K_0(B.GL_{N/S})$, for all $N \geq 1$.

- (ii) Assume that \mathbb{F} is weakly equivalent to $\mathcal{K}(\mathcal{F}(*))$, where $\mathcal{K}(\cdot)$ is the sheaf version of the Dold-Puppe functor, and $\mathcal{F}(*)$ is a graded sheaf giving a twisted duality cohomology theory in the sense of Gillet ([Gil81]). Let $\Phi, \Phi' : S.\mathcal{P} \rightarrow \mathbb{F}$ be two H -space maps. Then, the morphisms

$$\Phi, \Phi' : K_m(X) \rightarrow H^*(X, \mathcal{F}(*)), \quad m \geq 0$$

agree for all schemes X in \mathbf{C} , if they agree over $K_0(Gr_S(N, k))$, for all N, k .

It follows from (i) that there is a unique way to extend the Adams operations from the Grothendieck of vector bundles over a scheme X to higher K -theory by means of a sheaf map (in the homotopy category of simplicial sheaves). Analogously, the result (ii) implies that there is a unique way to extend the Chern character of vector bundles over a scheme X to higher degrees by means of a sheaf map (in the homotopy category of simplicial sheaves).

The paper is organized as follows. The first two sections are dedicated to review part of the theory developed by Gillet and Soulé in [GS99]. More concretely, in Section 1 we recall the main concepts about the homotopy theory of simplicial sheaves and generalized cohomology theories. In Section 2 we explain how K -theory can be given in this setting. We introduce the simplicial sheaves $\mathbb{K}^N = \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_N$ and $\mathbb{K} = \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL$.

In Section 3, we consider compatible systems of maps $\Phi_N : \mathbb{K}^N \rightarrow \mathbb{F}$, for $N \geq 1$ and \mathbb{F} a simplicial sheaf. We introduce the class of *weakly additive* system of maps, which are the ones characterized in this paper. Roughly speaking, they are the systems for which all the information can be obtained separately from the composition $\mathbb{Z}_{\infty} B.GL_N \hookrightarrow \mathbb{K}^N \rightarrow \mathbb{F}$ and from the composition $\mathbb{Z} \hookrightarrow \mathbb{K}^N \rightarrow \mathbb{F}$. They are named *weakly additive* due to the fact that when \mathbb{F} is an H -space, they are the maps given by the sum of this two mentioned compositions. The main example are the systems of the type $\mathbb{K}^N \hookrightarrow \mathbb{K} \xrightarrow{\Phi} \mathbb{F}$, inducing group morphisms on cohomology. We discuss then the comparison of two different weakly additive systems of maps. We end this section by applying the general discussion to generalized cohomology theories on a Zariski site.

In the last two sections we develop the application of the characterization results to K -theory and to cohomology theories. Section 4 is devoted to maps from K -theory to K -theory, concretely to the Adams and lambda operations on higher algebraic K -theory. A characterization of these

operations is given and the comparison to Grayson Adams operations is provided. In section 5, we consider maps from the K -groups to suitable sheaf cohomologies. We give a characterization of the Chern character and of the Chern classes for the higher algebraic K -groups of a scheme.

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1. THE HOMOTOPY CATEGORY OF SIMPLICIAL SHEAVES

We review here the main definitions and properties of homotopy theory of simplicial sheaves. For more details about this topic see [GS99]. For general facts and definitions about model categories we refer for instance to [Hir03].

Let \mathbf{C} be a site and let $\mathbf{T} = T(\mathbf{C})$ be the (Grothendieck) topos of sheaves on \mathbf{C} . We will always suppose that \mathbf{T} has enough points (see [SGA73], § IV 6.4.1).

Let \mathbf{sT} be the category of simplicial objects in \mathbf{T} . One identifies \mathbf{sT} with the category of sheaves of simplicial sets on \mathbf{C} . An object of \mathbf{sT} is called a *space*.

1.1. Structure of simplicial model category. The category \mathbf{sT} is endowed with a structure of simplicial model category in the sense of Quillen [Qui67]. This result is due to Joyal; a proof of it can be found in [Jar87], corollary. 2.7. Here we recall the definitions that give a simplicial model structure to \mathbf{sT} .

The structure of *model category* of \mathbf{sT} is given as follows. Let $X.$ be a space in \mathbf{sT} . One defines $\pi_0(X.)$ to be the sheaf associated to the presheaf

$$U \mapsto \pi_0(X.(U)), \quad \text{for } U \in \text{Obj}(\mathbf{C}).$$

Let $\mathbf{C}|U$ be the site of objects over U as described in [SGA73], § III 5.1, and let $\mathbf{T}|U$ denote the corresponding topos. For every object $X.$ in \mathbf{sT} , let $X.|U$ be the restriction of $X.$ to $\mathbf{sT}|U$. Then, for every $U \in \text{Obj}(\mathbf{C})$, $x \in X_0(U)$ a vertex of the simplicial set $X.(U)$, and every integer $n > 0$, one defines $\pi_n(X.|U, x)$ to be the sheaf associated to the presheaf

$$V \mapsto \pi_n(X.(V), x), \quad \text{for } V \in \text{Obj}(\mathbf{C}|U).$$

Let $X., Y.$ be two spaces and let $f : X. \rightarrow Y.$ be a map.

- (i) The map f is called a *weak equivalence* if the induced map $f_* : \pi_0(X.) \rightarrow \pi_0(Y.)$ is an isomorphism and, for all $n > 0$, $U \in \text{Obj}(\mathbf{C})$ and $x \in X_0(U)$, the natural maps

$$f_* : \pi_n(X.|U, x) \rightarrow \pi_n(Y.|U, f(x))$$

are isomorphisms.

- (ii) The map f is called a *cofibration* if for every $U \in \text{Obj}(\mathbf{C})$, the induced map

$$f(U) : X.(U) \rightarrow Y.(U)$$

is a cofibration of simplicial sets, i.e. it is a monomorphism.

- (iii) The map f is called a *fibration* if it has the right lifting property with respect to trivial cofibrations.

Observe that since the only map $\emptyset \rightarrow X.$ is always a monomorphism, all objects $X.$ in \mathbf{sT} are cofibrant.

Let \mathbf{SSets} denote the category of simplicial sets. The structure of *simplicial category* of \mathbf{sT} is given by the following definitions:

- (i) There is a functor $\mathbf{SSets} \rightarrow \mathbf{sT}$, which sends every simplicial set $K.$ to the sheafification of the constant presheaf that takes the value $K.$ for every U in \mathbf{C} .

- (ii) For every space X and every simplicial set K , the direct product $X \times K$ in \mathbf{sT} is the simplicial set given by

$$[n] \mapsto \coprod_{\sigma \in K_n} X_n,$$

and induced face and degeneracy maps.

- (iii) Let X, Y be two spaces and let Δ^n be the standard n -simplex in \mathbf{SSets} . The simplicial set $\underline{\mathbf{Hom}}(X, Y)$ is the functor

$$[n] \mapsto \mathbf{Hom}_{\mathbf{sT}}(X \times \Delta^n, Y).$$

Note that by definition, a map is a cofibration of spaces if and only if it is a section-wise cofibration of simplicial sets. For fibrations and weak equivalences, this is not always true. However, it follows from the definition that a section-wise weak equivalence is a weak equivalence of spaces.

1.2. Fibrant resolutions and the homotopy category. Let \mathcal{C} be any model category and $f : X \rightarrow Y$ a map. By definition, there exist two factorizations (α, β) of f ,

- (1) $f = \beta \circ \alpha$, with α a cofibration and β is a trivial fibration,
- (2) $f = \delta \circ \gamma$, where γ is a trivial cofibration and δ is a fibration.

A *fibrant resolution* of an object X , is a fibrant object X^\sim together with a trivial cofibration $X \xrightarrow{\sigma} X^\sim$. *Cofibrant resolutions* are defined dually. Its existence is guaranteed by the existence of the factorizations above.

One can form the *homotopy category* $\mathbf{Ho}(\mathbf{sT})$, associated to \mathbf{sT} , by formally inverting the weak equivalences. For any two spaces X, Y , one denotes by $[X, Y]$ the set of maps between X and Y in this category.

If $Y \rightarrow Y^\sim$ is any fibrant resolution of Y , then

$$[X, Y] = \pi_0 \mathbf{Hom}(X, Y^\sim).$$

More generally, suppose that $Y \rightarrow \hat{Y}$ is a weak equivalence (not necessarily also a cofibration) and \hat{Y} is fibrant. Then, if Y^\sim is any fibrant resolution, there exists a weak equivalence $Y^\sim \rightarrow \hat{Y}$. Therefore, by [Hir03], 9.5.12,

$$[X, Y] = \pi_0 \mathbf{Hom}(X, \hat{Y}).$$

Consider $X, Y \in \mathbf{Obj}(\mathcal{C})$ and $f : X \rightarrow Y$ a morphism. Suppose that Y is fibrant and let X^\sim be a fibrant resolution of X . Then, f factors uniquely (up to homotopy) through X^\sim , i.e. there exists a map in \mathcal{C} , $f^\sim : X^\sim \rightarrow Y$, unique up to homotopy under X , such that the following diagram is commutative

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \sigma \downarrow \sim & \nearrow f^\sim & \\ X^\sim & & \end{array}$$

See [Hir03], 8.1.6 for a proof. Therefore, there is a map

$$[X, Y] \rightarrow [X^\sim, Y]$$

obtained by the factorizations.

1.3. The category of simplicial presheaves. Let $\mathbf{sPre}(\mathbf{C})$ be the category of simplicial presheaves on \mathbf{C} , i.e. the category of functors $\mathbf{C}^{op} \rightarrow \mathbf{SSets}$. Then, one defines:

- ▷ *weak equivalences* and *cofibrations* of simplicial presheaves exactly as for simplicial sheaves, and,
- ▷ *fibrations* to be the maps satisfying the right lifting property with respect to trivial cofibrations.

As shown by Jardine in [Jar87], these definitions equip $\mathbf{sPre}(\mathbf{C})$ with a model category structure. The sheafification functor

$$\mathbf{sPre}(\mathbf{C}) \xrightarrow{(\cdot)^s} \mathbf{sT}$$

induces an equivalence between the respective homotopy categories, sending weak equivalences to weak equivalences. Moreover, the natural map $X. \rightarrow X.^s$ is a weak equivalence of simplicial presheaves ([Jar87], Lemma 2.6).

1.4. The category of pointed simplicial sheaves. Let \mathbf{sT}_* denote the category of pointed simplicial sheaves with morphisms being the maps preserving the base points. The definitions and results stated above can be translated into this category by considering the analogous definitions for pointed objects (see [HW98] §B.1 for details).

Let $*$ be the base point of \mathbf{sT}_* (i.e., the final and initial object). Then, if $X.$ is a simplicial sheaf, one considers its associated pointed object to be $X._+ = X. \sqcup *$.

1.5. Generalized cohomology theories. Let $X.$ be any space in \mathbf{sT}_* . The *suspension* of $X.$, $S \wedge X.$, is defined to be the space $X. \wedge \Delta^1 / \sim$, where \sim is the equivalence relation generated by $(x, 0) \sim (x, 1)$ and where \wedge is the pointed product. The *loop space functor* Ω is the right adjoint functor of S in the homotopy category.

Let $A.$ be any space in \mathbf{sT}_* . For every space $X.$ in \mathbf{sT}_* , one defines the *cohomology of $X.$ with coefficients in $A.$* as

$$H^{-m}(X., A.) = [S^m \wedge X., A.], \quad m \geq 0.$$

This is a pointed set for $m = 0$, a group for $m > 0$ and an abelian group for $m > 1$.

An infinite loop spectrum A^* is a collection of spaces $\{A^i\}_{i \geq 0}$, together with given weak equivalences $A^i \xrightarrow{\sim} \Omega A^{i+1}$. The *cohomology with coefficients in the spectrum A^** is defined as

$$H^{n-m}(X., A^*) = [S^m \wedge X., A^n], \quad m, n \geq 0.$$

Due to the adjointness relation between the loop space functor and the suspension, these sets depend only on the difference $n - m$. Therefore, all of them are abelian groups.

Let $A.$ be a simplicial sheaf and assume that there is an infinite loop spectrum A^* with $A^0 = A.$. Then the cohomology groups with coefficients in $A.$ are also defined with positive indices, with respect to this infinite loop spectrum. By abuse of notation, when there is no source of confusion, we will write $H^m(X., A.)$, for the generalized cohomology with positive indices, instead of writing $H^m(X., A^*)$.

When $X.$ is a non-pointed space in \mathbf{sT} , we define

$$H^{-m}(X., A.) = [S^m \wedge X._+, A.], \quad m \geq 0.$$

1.6. Induced morphisms. Let $A., B.$ be two pointed spaces. Every element $f \in [A., B.]$ induces functorial maps

$$[X., A.] \xrightarrow{f_*} [X., B.] \quad \text{and} \quad [B., X.] \xrightarrow{f^*} [A., X.],$$

for every space $X.$. Therefore, there are induced maps between the generalized cohomology groups

$$H^{-*}(X., A.) \xrightarrow{f_*} H^{-*}(X., B.) \quad \text{and} \quad H^{-*}(B., X.) \xrightarrow{f^*} H^{-*}(A., X.).$$

Using simplicial resolutions, these maps can be described as follows. If B^\sim is any fibrant resolution of B , then f is given by a homotopy class of maps $A \rightarrow B^\sim$. This map factorizes, uniquely up to homotopy, through a fibrant resolution of A , A^\sim . Therefore there is a map

$$f^\sim : A^\sim \rightarrow B^\sim$$

which induces, for every $m \geq 0$, a map

$$H^{-m}(X, A) = \pi_0 \operatorname{Hom}(S^m \wedge X, A^\sim) \rightarrow \pi_0 \operatorname{Hom}(S^m \wedge X, B^\sim) = H^{-m}(X, B).$$

The description of f^* is analogous.

1.7. Zariski topos. By the *big Zariski site*, \mathbf{ZAR} , we refer to the category of all noetherian schemes of finite Krull dimension, equipped with the Zariski topology.

Given any scheme X , one can consider the category formed by the inclusion maps $V \rightarrow U$ with U and V open subsets of X and then define the covers of $U \subseteq X$ to be the open covers of U . This is called the *small Zariski site* of X , $\mathbf{Zar}(X)$. By the *big Zariski site* of X , $\mathbf{ZAR}(X)$, we mean the category of all schemes of finite type over X equipped with the Zariski topology.

The corresponding topos are named the small or big Zariski topos (over X) respectively.

Generally, one also considers subsites of the big and small Zariski sites. For instance, the site of all noetherian schemes of finite Krull dimension which are also smooth (regular, quasi-projective or projective resp.) is a subsite of \mathbf{ZAR} . Similar subsites can be defined in $\mathbf{ZAR}(X)$ and $\mathbf{Zar}(X)$, depending on the properties of X .

At any of these sites, one associates to every scheme X in the underlying category \mathbf{C} , the constant pointed simplicial sheaf

$$U \mapsto \operatorname{Map}_{\mathbf{C}}(U, X) \cup \{*\}, \quad U \in \operatorname{Obj} \mathbf{C}.$$

This simplicial sheaf is also denoted by X . For any simplicial sheaf \mathbb{F} and any scheme X in \mathbf{C} , the equality of simplicial sets

$$\mathbb{F}(X) = \operatorname{Hom}(X, \mathbb{F})$$

is satisfied.

Definition 1.1. A space X is said to be *constructed from schemes* if, for all $n \geq 0$, X_n is representable by a scheme in the site plus a disjoint base point. If P is a property of schemes, one says that X satisfies the property P , if this is the case for the scheme parts of the components.

Any simplicial scheme gives rise to a space constructed from schemes, but the converse is not true (see [HW98], §B.1).

If X is a space constructed from schemes, we can write $X_n = * \coprod X'_n$, with X'_n a scheme. For every pointed simplicial sheaf \mathbb{F} in \mathbf{sT}_* , set $\mathbb{F}(X_n) := \mathbb{F}(X'_n)$. Then, one defines

$$\mathbb{F}(X) = \varprojlim_n \mathbb{F}(X_n),$$

where holim is the homotopy limit functor defined in [BK72].

1.8. Pseudo-flasque presheaves. We fix \mathbf{T} to be the topos associated to any Zariski site \mathbf{C} as in the previous section. The next definition is at the end of §2 in [BG73].

Definition 1.2. Let \mathbb{F} be a pointed simplicial presheaf on \mathbf{C} . It is called a *pseudo-flasque presheaf*, if the following two conditions hold:

- (i) $\mathbb{F}(\emptyset) = 0$.

(ii) For every pair of open subsets U, V of some scheme X , the square

$$\begin{array}{ccc} \mathbb{F}.(U \cap V) & \longrightarrow & \mathbb{F}.(V) \\ \downarrow & & \downarrow \\ \mathbb{F}.(U) & \longrightarrow & \mathbb{F}.(U \cup V) \end{array}$$

is homotopy cartesian.

A pseudo-flasque presheaf $\mathbb{F}.$ satisfies the Mayer-Vietoris property, i.e. for any scheme X in the site \mathbf{C} and any two open subsets U, V of X , there is a long exact sequence

$$\dots \rightarrow H^i(\mathbb{F}.(U \cup V)) \rightarrow H^i(\mathbb{F}.(U) \oplus \mathbb{F}.(V)) \rightarrow H^i(\mathbb{F}.(U \cap V)) \rightarrow H^{i+1}(\mathbb{F}.(U \cup V)) \rightarrow \dots$$

The importance of pseudo-flasque presheaves relies on the following proposition, due to Brown and Gestern (see [BG73], Theorem 4).

Proposition 1.3. *Let $\mathbb{F}.$ be a pseudo-flasque presheaf. For every scheme X in \mathbf{C} , the natural map*

$$\pi_i(\mathbb{F}.(X)) \rightarrow H^{-i}(X, \mathbb{F}^s)$$

is an isomorphism.

Observe that this proposition is already true for any fibrant space. In fact, any fibrant space is pseudo-flasque.

2. K-THEORY AS A GENERALIZED COHOMOLOGY

Let $X.$ be a space such that its 0-skeleton is reduced to one point. One defines $\mathbb{Z}_\infty X.$ to be the sheaf associated to the presheaf

$$U \mapsto \mathbb{Z}_\infty X.(U),$$

the functor \mathbb{Z}_∞ being the Bousfield-Kan integral completion of [BK72], § I. It comes equipped with a natural map $X. \rightarrow \mathbb{Z}_\infty X.$

Following [GS99], §3.1, we consider $(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$ a ringed topos with $\mathcal{O}_{\mathbf{T}}$ unitary and commutative. Then, for any integer $N \geq 1$, the linear group of rank N in \mathbf{T} , $GL_N = GL_N(\mathcal{O}_{\mathbf{T}})$, is the sheaf associated to the presheaf

$$U \mapsto GL_N(\Gamma(U, \mathcal{O}_{\mathbf{T}})).$$

Let $B.GL_N = B.GL_N(\mathcal{O}_{\mathbf{T}})$ be the classifying space of this sheaf of groups. Observe that for every $N \geq 1$, there is a natural inclusion $B.GL_N \hookrightarrow B.GL_{N+1}$. Consider the space $B.GL = \bigcup_N B.GL_N$ and the following pointed spaces

$$\begin{aligned} \mathbb{K}. &= \mathbb{Z} \times \mathbb{Z}_\infty B.GL, \\ \mathbb{K}.^N &= \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N. \end{aligned}$$

Here, \mathbb{Z} is the constant simplicial sheaf given by the constant sheaf \mathbb{Z} , pointed by zero. For every $N \geq 1$, the direct sum of matrices together with addition over \mathbb{Z} gives a map

$$\mathbb{K}.^N \wedge \mathbb{K}.^N \rightarrow \mathbb{K}..$$

These maps are compatible with the natural inclusions; thus $\mathbb{K}.$ is equipped with an H-space structure (see [HW98]).

2.1. K-theory. Following [GS99], for any space $X.$, the *stable K-theory* is defined as

$$H^{-m}(X., \mathbb{K}.) = [S^m \wedge X._+, \mathbb{K}.],$$

and for every $N \geq 1$, the *unstable K-theory* is defined as

$$H^{-m}(X., \mathbb{K}.^N) = [S^m \wedge X._+, \mathbb{K}.^N].$$

Since $\mathbb{K}.$ is an H -space, $H^{-m}(X., \mathbb{K}.)$ are abelian groups for all m . However, $H^{-m}(X., \mathbb{K}.^N)$ are abelian groups for all $m > 0$ and in general only pointed sets for $m = 0$.

Definition 2.1. A space $X.$ is *K-coherent* if the natural maps

$$\lim_{\overrightarrow{N}} H^{-m}(X., \mathbb{K}.^N) \rightarrow H^{-m}(X., \mathbb{K}.)$$

and

$$\lim_{\overrightarrow{N}} H^m(X., \pi_n \mathbb{K}.^N) \rightarrow H^m(X., \pi_n \mathbb{K}.)$$

are isomorphisms for all $m, n \geq 0$.

(Here $H^m(X., \pi_n Y)$ are the singular cohomology groups. See [GS99], §1.2 for a discussion in this language).

The Loday product induces a product structure on $H^{-*}(X., \mathbb{K}.)$ for every K -coherent space $X.$.

2.2. Comparison to Quillen's K-theory. Let $(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$ be a locally ringed topos. For every U in \mathbf{T} , let $\mathcal{P}(U)$ be the category of locally free $\mathcal{O}_{\mathbf{T}|U}$ -sheaves of finite rank.

Let $B.Q\mathcal{P}$ be the simplicial sheaf obtained by the Quillen construction applied to every $\mathcal{P}(U)$ (see [Qui73]). If $\Omega B.Q\mathcal{P}$ is the loop space of $B.Q\mathcal{P}$, then, by the results of [GS99], §3.2.1 and [Gil81], Proposition 2.15, we obtain:

Lemma 2.2. *In the homotopy category of simplicial sheaves, there is a natural map of spaces*

$$\mathbb{Z} \times \mathbb{Z}_{\infty} B.GL \rightarrow \Omega B.Q\mathcal{P}$$

which is a weak equivalence.

Observe that this means that $\mathbb{K}.$ has yet another H -space structure, given by the Waldhausen's pairing [Wal78] $\Omega B.Q\mathcal{P} \wedge \Omega B.Q\mathcal{P} \rightarrow \Omega B.Q\mathcal{P}$. As stated in [GS99] §3.2.1, both structures agree.

It follows from the lemma that for any space $X.$ in \mathbf{sT} , there is an isomorphism

$$H^{-m}(X., \mathbb{K}.) \cong H^{-m}(X., \Omega B.Q\mathcal{P}).$$

Hence, the stable K -theory of a space can be computed using the simplicial sheaf $\Omega B.Q\mathcal{P}$ instead of the simplicial sheaf $\mathbb{K}.$.

Suppose that \mathbf{T} is the category of sheaves over a category of schemes \mathbf{C} . Let \mathbb{K}^{\sim} be a fibrant resolution of $\Omega B.Q\mathcal{P}$. For every scheme X in \mathbf{C} , there is a natural map

$$(2.3) \quad K_m(X) = \pi_m(\Omega B.Q\mathcal{P}(X)) \rightarrow \pi_m(\mathbb{K}^{\sim}(X)) \cong H^{-m}(X, \mathbb{K}.).$$

The next theorem shows that many schemes are K -coherent and that Quillen K -theory agrees with stable K -theory.

Theorem 2.4 ([GS99], Proposition 5). *Suppose that X is a noetherian scheme of finite Krull dimension d and that \mathbf{T} is either*

- (1) ZAR , the big Zariski site of all noetherian schemes of finite Krull dimension,
- (2) $\text{ZAR}(X)$, the big Zariski site of all schemes of finite type over X ,
- (3) $\text{Zar}(X)$, the small Zariski site of X .

Then, viewed as a \mathbf{T} -space, X is K -coherent with cohomological dimension at most d . Furthermore, the morphisms $K_m(X) \rightarrow H^{-m}(X, \mathbb{K}_.)$ are isomorphisms for all m .

Remark 2.5. Let \mathcal{C} be a small category of schemes over X that contains all open subschemes of its objects. Consider the subsite $Z(X)$ of $\text{ZAR}(X)$ obtained by endowing \mathcal{C} with the Zariski topology. Then, the statement of the theorem will be true with $\mathbf{T} = T(Z(X))$.

For instance, if X is a regular noetherian scheme of finite Krull dimension, we could consider $Z(X)$ to be the site of all regular schemes of finite type over X as $Z(X)$. Another example would be the site of all quasi-projective schemes of finite type over a noetherian quasi-projective scheme of finite Krull dimension.

2.3. K-theory of spaces constructed from schemes. Let $\mathbf{C} = \text{ZAR}$ and let X be a space constructed from schemes. Then, in the Quillen context, one defines

$$K_m(X) = \pi_{m+1}(\varprojlim_n B.Q\mathcal{P}(X_n)).$$

For a description of the functor holim , see [BK72], § XI, for the case of simplicial sets or see [Hir03], §19 for a general treatment.

Observe that the construction of the map (2.3) can be extended to spaces constructed from schemes. A space X is said to be *degenerate* (above some simplicial degree) if there exists an $N \geq 0$ such that $X = \text{sk}_N X$. (where sk_N means the N -th skeleton of X).

The next proposition is found in [GS99], §3.2.3.

Proposition 2.6. *Let X be a space constructed from schemes in ZAR . Then, the morphism (2.3) gives an isomorphism $K_m(X) \cong H^{-m}(X, \mathbb{K}_.)$. Moreover, if X is degenerate, then X is K -coherent.*

In particular, in the big Zariski site, since for every $N \geq 1$, $B.GL_N$ is a simplicial scheme, we have $K_m(B.GL_N) = H^{-m}(B.GL_N, \mathbb{K}_.)$. However, $B.GL_N$ is not degenerate.

In a Zariski site over a base scheme S , the simplicial sheaf $B.GL_N$ is the simplicial scheme given by the fibred product $B.GL_{N/S} = B.GL_N \times_{\mathbb{Z}} S$.

3. CHARACTERIZATION OF MAPS FROM K-THEORY

Our aim is to characterize functorial maps from K -theory. Since stable K -theory is expressed as a representable functor, a first approximation is obviously given by Yoneda's lemma. That is, given a space \mathbb{F} and a map of spaces $\mathbb{K} \xrightarrow{\Phi} \mathbb{F}$, the induced maps

$$H^{-m}(X, \mathbb{K}_.) \xrightarrow{\Phi_*} H^{-m}(X, \mathbb{F}_.), \quad \forall m \geq 0,$$

are determined by the image of $id \in [\mathbb{K}, \mathbb{K}]$ by the map $\Phi_* : H^0(\mathbb{K}, \mathbb{K}_.) \rightarrow H^0(\mathbb{K}, \mathbb{F}_.)$. Indeed, if $g \in H^{-m}(X, \mathbb{K}_.) = [S^m \wedge X, \mathbb{K}]$, there are induced morphisms

$$[\mathbb{K}, \mathbb{K}] \xrightarrow{g^*} [S^m \wedge X, \mathbb{K}] \quad \text{and} \quad [\mathbb{K}, \mathbb{F}] \xrightarrow{g^*} [S^m \wedge X, \mathbb{F}].$$

Then, $g = g^*(id)$ and

$$\Phi_*(g) = \Phi_*g^*(id) = g^*\Phi_*(id).$$

We will see that, under some favorable conditions, the element id can be changed by other universal elements at the level of the simplicial scheme $B.GL_N$, for all $N \geq 1$.

3.1. Compatible systems of maps and Yoneda lemma. As in section 2, let $(\mathbf{T}, \mathcal{O}_{\mathbf{T}})$ be a ringed topos and let $\mathbb{F}.$ be a fibrant space in \mathbf{sT}_* . A system of maps $\Phi_M \in [\mathbb{K}^M, \mathbb{F}.]$, $M \geq 1$, is said to be *compatible* if, for all $M' \geq M$, the diagram

$$\begin{array}{ccc} \mathbb{K}^M & & \\ \downarrow & \searrow \Phi_M & \\ \mathbb{K}^{M'} & \xrightarrow{\Phi_{M'}} & \mathbb{F}. \end{array}$$

is commutative in $\mathrm{Ho}(\mathbf{sT}_*)$. We associate to any map $\Phi : \mathbb{K} \rightarrow \mathbb{F}.$ in $\mathrm{Ho}(\mathbf{sT}_*)$, a compatible system of maps $\{\Phi_M\}_{M \geq 1}$, given by the composition of Φ with the natural map from \mathbb{K}^M into $\mathbb{K}.$

Every compatible system of maps $\{\Phi_M\}_{M \geq 1}$ induces a natural transformation of functors

$$\Phi(-) : \lim_{\vec{M}} [-, \mathbb{K}^M] \rightarrow [-, \mathbb{F}].$$

We state here a variant of Yoneda's lemma for maps induced by a compatible system as above.

Lemma 3.1. *Let $\mathbb{F}.$ be a fibrant space in \mathbf{sT}_* . The map*

$$\left\{ \begin{array}{l} \text{compatible systems of maps} \\ \{\Phi_M\}_{M \geq 1}, \Phi_M \in [\mathbb{K}^M, \mathbb{F}.] \end{array} \right\} \xrightarrow{\alpha} \left\{ \begin{array}{l} \text{natural transformation of functors} \\ \Phi(-) : \lim_{\vec{M}} [-, \mathbb{K}^M] \rightarrow [-, \mathbb{F}.] \end{array} \right\}$$

sending every compatible system of maps to its induced natural transformation, is a bijection.

Proof. We prove the result by giving the explicit inverse arrow β of α .

So, let

$$\Phi(-) : \lim_{\vec{M}} [-, \mathbb{K}^M] \rightarrow [-, \mathbb{F}.]$$

be a natural transformation of functors. For every $N \geq 1$, let $e_N \in \lim_{\vec{M}} [\mathbb{K}^N, \mathbb{K}^M]$ be the image of $id \in [\mathbb{K}^N, \mathbb{K}^N]$ under the natural morphism $[\mathbb{K}^N, \mathbb{K}^N] \xrightarrow{\sigma_N} \lim_{\vec{M}} [\mathbb{K}^N, \mathbb{K}^M]$. We define $\Phi_N = \beta(\Phi)_N \in [\mathbb{K}^N, \mathbb{F}.]$ to be the image of e_N by Φ ,

$$\Phi_N := \Phi(\mathbb{K}^N)(e_N).$$

In general, for every $N' \geq N \geq 1$, consider the map $e_{N,N'} \in [\mathbb{K}^N, \mathbb{K}^{N'}]$ induced by the natural inclusion $B.GL_N \hookrightarrow B.GL_{N'}$. Observe that the image of $e_{N,N'}$ under the map

$$[\mathbb{K}^N, \mathbb{K}^{N'}] \xrightarrow{\sigma_{N'}} \lim_{\vec{M}} [\mathbb{K}^N, \mathbb{K}^M],$$

is exactly e_N . Moreover, by hypothesis, there is a commutative diagram

$$\begin{array}{ccc} \lim_{\vec{M}} [\mathbb{K}^{N'}, \mathbb{K}^M] & \xrightarrow{\Phi(\mathbb{K}^{N'})} & [\mathbb{K}^{N'}, \mathbb{F}.] \\ e_{N,N'}^* \downarrow & & \downarrow e_{N,N'}^* \\ \lim_{\vec{M}} [\mathbb{K}^N, \mathbb{K}^M] & \xrightarrow{\Phi(\mathbb{K}^N)} & [\mathbb{K}^N, \mathbb{F}.] \end{array}$$

which gives the compatibility of the system $\{\Phi_N\}_{N \geq 1}$. Therefore, the map β is defined.

Now let $X.$ be any space in \mathbf{sT}_* . In order to prove that β is a right inverse of α , we have to see that $\Phi(X.)$ is the map induced by the just constructed system $\{\Phi_M\}_{M \geq 1}$.

Let $f \in \lim_{\rightarrow M} [X., \mathbb{K}^M]$. Then, there exists an integer $N \geq 1$ and a map $g \in [X., \mathbb{K}^N]$, such that $\sigma_N(g) = f$. By the commutative diagram

$$\begin{array}{ccc} [\mathbb{K}^N, \mathbb{K}^N] & \xrightarrow{\sigma_N} & \lim_{\rightarrow M} [\mathbb{K}^N, \mathbb{K}^M] \\ g^* \downarrow & & \downarrow g^* \\ [X., \mathbb{K}^N] & \xrightarrow{\sigma_N} & \lim_{\rightarrow M} [X., \mathbb{K}^M], \end{array}$$

we see that in fact, $f = \sigma_N(g) = g^*(e_N)$. Using the fact that Φ is a natural transformation, the diagram

$$(3.2) \quad \begin{array}{ccc} \lim_{\rightarrow M} [\mathbb{K}^N, \mathbb{K}^M] & \xrightarrow{\Phi} & [\mathbb{K}^N, \mathbb{F}.] \\ g^* \downarrow & & \downarrow g^* \\ \lim_{\rightarrow M} [X., \mathbb{K}^M] & \xrightarrow{\Phi} & [X., \mathbb{F}.] \end{array}$$

is commutative. Hence, we obtain

$$\Phi(f) = \Phi(g^*(e_N)) = g^*\Phi(e_N) = \beta(\Phi)_N \circ g = \alpha\beta(\Phi)(f),$$

as desired.

It remains to check that β is a left inverse of α . Let $\{\Phi_N\}_{N \geq 1}$ be a compatible system of maps, let Φ be the associated transformation of functors obtained by α and let $\{\beta(\Phi)_N\}_{N \geq 1}$ be the system $\beta(\Phi)$. From the commutative diagram

$$\begin{array}{ccc} [\mathbb{K}^N, \mathbb{K}^N] & \xrightarrow{(\Phi_N)_*} & [\mathbb{K}^N, \mathbb{F}.] \\ \sigma_N \downarrow & \nearrow \Phi & \\ \lim_{\rightarrow M} [\mathbb{K}^N, \mathbb{K}^M] & & \end{array}$$

we deduce that

$$\Phi_N = (\Phi_N)_*(id) = \Phi\sigma_N(id) = \Phi(e_N) = \beta(\Phi)_N.$$

Therefore, β is the inverse of α and thus α is a bijection. \square

Remark 3.3. The last lemma is not specific to our category and to our compatible system of maps. It could be directly generalized to any suitable category.

3.2. Weakly additive systems of maps. We start by defining the class of weakly additive systems of maps. It is for this class of maps that we will state our results on the comparison of the induced maps. It will be shown below that many usual maps are weakly additive.

Let $\mathbb{K}^N, \mathbb{K}.$ be as in the previous section. Let pr_1 and pr_2 be the projections onto the first and second component respectively

$$\begin{aligned} \text{pr}_1 &: \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_M \rightarrow \mathbb{Z}, \\ \text{pr}_2 &: \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_M \rightarrow \mathbb{Z}_{\infty} B.GL_M, \end{aligned}$$

and let j_1, j_2 denote the inclusions obtained using the respective base points

$$\begin{aligned} \mathbb{Z} &\xrightarrow{j_1} \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_M, \\ \mathbb{Z}_{\infty} B.GL_M &\xrightarrow{j_2} \mathbb{Z} \times \mathbb{Z}_{\infty} B.GL_M. \end{aligned}$$

Denote by $\pi_i = j_i \circ \text{pr}_i \in [\mathbb{K}^M, \mathbb{K}^M]$, $i = 1, 2$, the compositions

$$\begin{aligned} \pi_1 : \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M &\xrightarrow{\text{pr}_1} \mathbb{Z} \xrightarrow{j_1} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M, \\ \pi_2 : \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M &\xrightarrow{\text{pr}_2} \mathbb{Z}_\infty B.GL_M \xrightarrow{j_2} \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M. \end{aligned}$$

For every space \mathbb{F} . in \mathbf{sT}_* , there are induced maps

$$[\mathbb{K}^M, \mathbb{F}.] \xrightarrow{\pi_i^*} [\mathbb{K}^M, \mathbb{F}.], \quad i = 1, 2.$$

If $\Phi_M \in [\mathbb{K}^M, \mathbb{F}.]$ we define the maps

$$\Phi_M^i := \pi_i^*(\Phi_M) \in [\mathbb{K}^M, \mathbb{F}.], \quad i = 1, 2.$$

Remark 3.4. Consider a compatible system of maps $\Phi_M \in [\mathbb{K}^M, \mathbb{F}.]$, $M \geq 1$ and assume that \mathbb{F} . is fibrant (this is no loss of generality). By hypothesis, the diagrams

$$\begin{array}{ccc} & \mathbb{K}^M & \\ j_1 \nearrow & \downarrow & \\ \mathbb{Z} & \xrightarrow{j_1} & \mathbb{K}^{M'} \end{array} \quad \text{and} \quad \begin{array}{ccc} & \mathbb{K}^M & \\ \downarrow & \searrow \Phi_M & \\ \mathbb{K}^{M'} & \xrightarrow{\Phi_{M'}} & \mathbb{F}. \end{array}$$

are commutative in $\text{Ho}(\mathbf{sT}_*)$, for every $M' \geq M$. Therefore, the homotopy class of the map $\Phi_M \circ j_1$ in $[\mathbb{Z}, \mathbb{F}.]$ does not depend on M .

Definition 3.5. Let \mathbb{F} . be any space in \mathbf{sT}_* . By an operation $\bullet = \{\bullet_M\}_{M \geq 1}$ on $\{[\mathbb{K}^M, \mathbb{F}.]\}_{M \geq 1}$ we mean a collection of maps

$$\bullet_M : [\mathbb{K}^M, \mathbb{F}.] \times [\mathbb{K}^M, \mathbb{F}.] \rightarrow [\mathbb{K}^M, \mathbb{F}.], \quad M \geq 1.$$

Note that we do not require compatibility for different indices M .

Let $\{\Phi_M\}_{M \geq 1}$, with $\Phi_M \in [\mathbb{K}^M, \mathbb{F}.]$, be a compatible system of maps. The system $\{\Phi_M\}_{M \geq 1}$ is called *weakly additive with respect to an operation* $\bullet = \{\bullet_M\}_{M \geq 1}$, if for every M , there is an equality

$$\Phi_M = \Phi_M^1 \bullet_M \Phi_M^2.$$

Any map $\bullet : [\mathbb{K}., \mathbb{F}.] \wedge [\mathbb{K}., \mathbb{F}.] \rightarrow [\mathbb{K}., \mathbb{F}.]$ induces an operation on $\{[\mathbb{K}^M, \mathbb{F}.]\}_{M \geq 1}$ by composition with the natural maps $\mathbb{K}^M \rightarrow \mathbb{K}.$. Then, a map $\Phi \in [\mathbb{K}., \mathbb{F}.]$ is called *weakly additive* with respect to \bullet if the induced compatible system of maps is weakly additive with respect to the induced operation $\{\bullet_M\}_{M \geq 1}$.

Example 3.6 (Φ is trivial over \mathbb{Z}). Let $\{\Phi_M\}_{M \geq 1}$ be a compatible system of maps with $\Phi_M \in [\mathbb{K}^M, \mathbb{F}.]$. Assume that for all $M \geq 1$, $\Phi_M^1 = *$, i.e. the constant map to the base point of $\mathbb{F}.$. Then, $\Phi_M = \Phi_M^2$ for all M . Therefore, if we take the operation $\bullet_M = \text{pr}_2$, the system $\{\Phi_M\}_{M \geq 1}$ is weakly additive with respect to $\bullet = \{\bullet_M\}_{M \geq 1}$.

Example 3.7 (\mathbb{F} . is an H -space). Assume that \mathbb{F} . is an H -space. Then, one can take the operation \bullet_M to be the sum in $[\mathbb{K}^M, \mathbb{F}.]$. Then for a compatible system of maps $\{\Phi_M\}_{M \geq 1}$ the condition of weakly additivity means that the maps Φ_M behave additively over the two components of \mathbb{K}^M . Actually, the definition of weakly additive systems of maps was motivated by this example.

The lambda operations on higher algebraic K -theory, defined by Gillet and Soulé in [GS99], are an example of this type of weakly additive systems of maps.

Remark 3.8. If \mathbb{F} . is an H -space and $\Phi = \{\Phi_M\}_{M \geq 1}$ is a compatible system of maps such that $\Phi_M^1 = *$, then the system Φ is weakly additive with respect to the H -sum of \mathbb{F} . (previous example) and also with respect to the operation $\bullet_M = \text{pr}_2$ (by example 3.6).

3.3. Classifying elements. We now introduce some classifying elements. For every $N \geq 1$, let

$$\sigma_N : [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^N] \rightarrow \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M]$$

be the natural morphism. Recall that we defined $j_2 \in [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^N]$ to be the map induced by the natural inclusion. Then, we define

$$i'_N = \sigma_N(j_2) \in \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M].$$

Let $r \geq 0$ and $u'_r \in \varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M]$ be the image by σ_N of the homotopy class of the constant map

$$\begin{aligned} \mathbb{Z}_\infty B.GL_N &\rightarrow \mathbb{Z} \times \mathbb{Z}_\infty B.GL_N \\ x &\mapsto (r, *). \end{aligned}$$

Finally, consider the natural map $B.GL_N \rightarrow \mathbb{Z}_\infty B.GL_N$. The images of i'_N and u'_r under the induced map

$$\varinjlim_M [\mathbb{Z}_\infty B.GL_N, \mathbb{K}^M] \rightarrow \varinjlim_M [B.GL_N, \mathbb{K}^M],$$

are denoted by

$$i_N, u_r \in \varinjlim_M [B.GL_N, \mathbb{K}^M]$$

respectively.

Proposition 3.9. *Let \mathbb{F} be a space in \mathbf{sT}_* and let $\{\Phi_M\}_{M \geq 1}, \{\Phi'_M\}_{M \geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi, \Phi' : \varinjlim_M [-, \mathbb{K}^M] \rightarrow [-, \mathbb{F}]$$

agree for all spaces, if and only if, in $[\mathbb{Z}_\infty B.GL_N, \mathbb{F}]$ it holds

$$(3.10) \quad \Phi(i'_N) = \Phi'(i'_N), \quad \text{for all } N \geq 1,$$

$$(3.11) \quad \Phi(u'_r) = \Phi'(u'_r), \quad \text{for all } r \in \mathbb{Z}, N \geq 1.$$

Proof. One implication is obvious. By lemma 3.1, it is enough to see that for all N , $\Phi_N = \Phi'_N$. By hypothesis, there is an operation \bullet_N on $[\mathbb{K}^N, \mathbb{F}]$ such that

$$\begin{aligned} \Phi_N &= \Phi_N^1 \bullet_N \Phi_N^2, \\ \Phi'_N &= \Phi_N'^1 \bullet_N \Phi_N'^2. \end{aligned}$$

Therefore, it is enough to see that

$$\Phi_N^1 = \Phi_N'^1, \quad \text{and} \quad \Phi_N^2 = \Phi_N'^2.$$

The first equality follows from hypothesis (3.11). For the second equality, observe that by definition, $\Phi(i'_N) = \Phi_N \circ j_2$. Therefore, by equality (3.10),

$$\Phi_N^2 = \Phi_N \circ j_2 \circ \text{pr}_2 = \Phi(i'_N) \circ \text{pr}_2 = \Phi'(i'_N) \circ \text{pr}_2 = \Phi_N'^2.$$

□

Corollary 3.12. *Let \mathbb{F} be an H -space in \mathbf{sT}_* and $\{\Phi_M\}_{M \geq 1}, \{\Phi'_M\}_{M \geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi(-), \Phi'(-) : \varinjlim_M [-, \mathbb{K}^M] \rightarrow [-, \mathbb{F}]$$

agree if and only if, in $[B.GL_N, \mathbb{F}]$ it holds

$$(3.13) \quad \Phi(i_N) = \Phi'(i_N), \quad \text{for all } N \geq 1,$$

$$(3.14) \quad \Phi(u_r) = \Phi'(u_r), \quad \text{for all } r \in \mathbb{Z}, N \geq 1.$$

Proof. It follows from the fact that the natural map

$$[\mathbb{Z}_\infty B.GL_N, \mathbb{F}] \rightarrow [B.GL_N, \mathbb{F}]$$

is an isomorphism if \mathbb{F} is an H -space, and under this isomorphism, the elements u_r and i_N correspond to u'_r and i'_N respectively. \square

Let $j_N \in H^0(B.GL_N, \mathbb{K}) = K_0(B.GL_N)$ be the image of i_N under the morphism

$$\lim_{\overrightarrow{M}} H^0(B.GL_N, \mathbb{K}^M) \rightarrow H^0(B.GL_N, \mathbb{K}).$$

By abuse of notation, we denote by u_r the image of $u_r \in \lim_{\overrightarrow{M}} H^0(B.GL_N, \mathbb{K}^M)$ in $H^0(B.GL_N, \mathbb{K})$.

Corollary 3.15. *Let \mathbb{F} be an H -space in \mathbf{sT}_* . Let $\Phi, \Phi' : \mathbb{K} \rightarrow \mathbb{F}$ be two H -space maps. Then, Φ and Φ' are weakly additive. Moreover,*

$$\Phi, \Phi' : [X, \mathbb{K}] \rightarrow [X, \mathbb{F}],$$

agree for all K -coherent spaces X , if they agree at $j_N, u_r \in K_0(B.GL_N)$ for all $N \geq 1$ and all $r \in \mathbb{Z}$.

Proof. The maps Φ and Φ' are weakly additive with respect to the H -sum of \mathbb{F} , due to example 3.7. Therefore, by corollary 3.12, the maps

$$\Phi, \Phi' : [X, \mathbb{K}] \cong \lim_{\overrightarrow{N}} [X, \mathbb{K}^N] \rightarrow [X, \mathbb{F}]$$

agree for all K -coherent spaces X , if and only if $\Phi(i_N) = \Phi'(i_N)$ for $N \geq 1$ and $\Phi(u_r) = \Phi'(u_r)$ for all r . Since by construction $\Phi(i_N) = \Phi(j_N)$ (and the same for Φ'), the corollary is proved. \square

3.4. Application to the Zariski sites. Let S be a finite dimensional noetherian scheme. Fix \mathbf{C} a Zariski subsite of $\text{ZAR}(S)$ containing all open subschemes of its objects and the components of the simplicial scheme $B.GL_{N/S}$. Let $\mathbf{T} = T(\mathbf{C})$.

A direct consequence of corollary 3.12 is the following theorem.

Theorem 3.16. *Let \mathbb{F} be an H -space in \mathbf{sT}_* and $\{\Phi_M\}_{M \geq 1}, \{\Phi'_M\}_{M \geq 1}$ be two weakly additive systems of maps with respect to the same operation. Then, the induced maps*

$$\Phi, \Phi' : K_m(X) \cong \lim_{\overrightarrow{M}} H^{-m}(X, \mathbb{K}^M) \rightarrow H^{-m}(X, \mathbb{F})$$

agree for all $m \geq 0$ and all K -coherent spaces X , if and only if in $H^0(B.GL_{N/S}, \mathbb{F})$ it holds

$$\begin{aligned} \Phi(i_N) &= \Phi'(i_N), \quad \text{for all } N \geq 1, \\ \Phi(u_r) &= \Phi'(u_r), \quad \text{for all } r \in \mathbb{Z}, N \geq 1. \end{aligned}$$

\square

Finally, the next corollary is corollary 3.15 applied to the Zariski subsite $\mathbf{C} \subset \text{ZAR}(S)$.

Corollary 3.17. *Let \mathbb{F} be an H -space in \mathbf{sT}_* . Let $\chi_1, \chi_2 \in [\mathbb{K}, \mathbb{F}]$ be two H -space maps in $\text{Ho}(\mathbf{sT}_*)$. Then, the induced maps*

$$\chi_1, \chi_2 : K_m(X) \rightarrow H^{-m}(X, \mathbb{F})$$

agree for all degenerate simplicial schemes in \mathbf{C} , if for all $N \geq 1$ and $r \in \mathbb{Z}$, they agree at $j_N, u_r \in K_0(B.GL_{N/S})$.

□

The next theorem shows that in fact a weaker condition is needed in order to obtain the uniqueness of maps.

For any scheme X and any simplicial sheaf \mathbb{F} , let \mathbb{F}_X denote the restriction of \mathbb{F} to the small Zariski site of X . In the next theorem, we write $[\cdot, \cdot]_{\mathbf{C}}$ for the maps in $\mathrm{Ho}(\mathbf{sT}(\mathbf{C})_*)$, for any site \mathbf{C} . If X is a scheme in \mathbf{C} , let $\mathbf{C}(X)$ be the subsite of $\mathrm{ZAR}(X)$ whose objects are in \mathbf{C} .

Theorem 3.18. *Let \mathbb{F} be a pseudo-flasque sheaf on \mathbf{sT}_* , which is an H -space. Assume that*

► *For every scheme X in \mathbf{C} , there are two H -space maps*

$$\chi_i(X) \in [\mathbb{K}_X, \mathbb{F}_X]_{\mathrm{ZAR}(X)}, \quad i = 1, 2.$$

► *For any map $X \rightarrow Y$ in \mathbf{C} and for $i = 1, 2$ there is a commutative diagram*

$$(3.19) \quad \begin{array}{ccc} \mathbb{K}_\bullet(Y) & \xrightarrow{\chi_i(Y)} & \mathbb{F}_\bullet(Y) \\ \downarrow & & \downarrow \\ \mathbb{K}_\bullet(X) & \xrightarrow{\chi_i(X)} & \mathbb{F}_\bullet(X) \end{array}$$

in $\mathrm{Ho}(\mathbf{SSets}_)$.*

Then, the maps

$$\chi_1, \chi_2 : K_m(X) \rightarrow H^{-m}(X, \mathbb{F}_\bullet)$$

agree for all schemes in \mathbf{C} if they agree at $j_N, u_r \in K_0(B.GL_{N/S})$, for all $N \geq 1$ and $r \in \mathbb{Z}$ (see remark below).

Remark 3.20. The condition of \mathbb{F} being pseudo-flasque means that for any space X constructed from schemes,

$$H^{-m}(X, \mathbb{F}_\bullet) \cong \pi_m(\mathbb{F}_\bullet(X)) := \pi_m(\varprojlim_n \mathbb{F}_\bullet(X_n)).$$

Now, the commutative diagram (3.19), implies that, for any such space, there are induced morphisms

$$\mathbb{K}_\bullet(X) \xrightarrow{\chi_i(X)} \mathbb{F}_\bullet(X), \quad i = 1, 2.$$

Hence, the maps χ_1 and χ_2 are defined for $X = B.GL_{N/S}$.

Proof. For every fixed scheme X , it follows from theorem 3.17 that $\chi_1(X) = \chi_2(X)$ if they agree at $j_N, u_r \in [B.GL_{N|X}, \mathbb{K}_X]_{\mathrm{ZAR}(X)}$. Observe now that by the remarks following proposition 2.4,

$$[B.GL_{N|X}, \mathbb{K}_X]_{\mathrm{ZAR}(X)} \cong [B.GL_{N|X}, \mathbb{K}_\bullet]_{\mathbf{C}(X)} = K_0(B.GL_{N|X}),$$

and for $i = 1, 2$ there is a commutative diagram

$$(3.21) \quad \begin{array}{ccc} K_0(B.GL_{N|S}) & \xrightarrow{\chi_i} & [B.GL_{N|S}, \mathbb{F}_\bullet]_{\mathbf{C}} \\ \downarrow & & \downarrow \\ K_0(B.GL_{N|X}) & \xrightarrow{\chi_i} & [B.GL_{N|X}, \mathbb{F}_\bullet]_{\mathbf{C}(X)} \end{array}$$

Then, the statement follows from the fact that j_N and u_r in $K_0(B.GL_{N|X})$ are the image under the vertical map of j_N and u_r in $K_0(B.GL_{N|S})$. □

4. MORPHISMS BETWEEN K-GROUPS

4.1. Lambda and Adams operations. In this section we focus on the case where $\mathbb{F} = \mathbb{K}$. Then, the main application of theorems 3.17 and 3.18 is to the Adams operations and to the lambda operations on higher algebraic K -theory.

The Grothendieck group of a scheme X , has a λ -ring structure given by $\lambda^k(E) = \bigwedge^k E$, for any vector bundle E over X . In the literature there are several definitions of the extension of the Adams operations of $K_0(X)$ to the higher algebraic K -groups. Our aim in this section is to give a criterion for their comparison.

Soulé, in [Sou85], gives a λ -ring structure to the higher algebraic K -groups of any noetherian regular scheme of finite Krull dimension. Gillet and Soulé then generalize this result in [GS99], defining lambda operations for all K -coherent spaces in any locally ringed topos. We briefly recall this construction here.

Let $\mathbf{R}_{\mathbb{Z}}(GL_N)$ be the Grothendieck group of representations of the general linear group scheme GL_N/\mathbb{Z} . The properties of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ that concern us are:

- (1) $\mathbf{R}_{\mathbb{Z}}(GL_N)$ has a λ -ring structure.
- (2) For any locally ringed topos, there is a ring morphism

$$\varphi : \mathbf{R}_{\mathbb{Z}}(GL_N) \rightarrow H^0(B.GL_N, \mathbb{K}).$$

The operations $\lambda_{GS}^k, \Psi_{GS}^k$ are constructed by transferring the lambda and Adams operations of $\mathbf{R}_{\mathbb{Z}}(GL_N)$ to the K -theory of $B.GL_N$. Namely, consider the representation $id_N - N$ and the maps

$$\varphi(\Psi^k(id_N - N)), \varphi(\lambda^k(id_N - N)) : B.GL_N \rightarrow \mathbb{K}.$$

in the homotopy category of simplicial sheaves.

Consider the only λ -ring structure on \mathbb{Z} with trivial involution. Then, adding the previous maps with the lambda or Adams operations on the \mathbb{Z} -component, we obtain compatible systems of maps

$$\Psi_{GS}^k, \lambda_{GS}^k : \mathbb{K}^N \rightarrow \mathbb{K}, \quad N \geq 1.$$

Observe that, by example 3.7, both systems are weakly additive with respect to the H -sum of \mathbb{K} .

In particular, for any noetherian scheme X of finite Krull dimension, there are induced Adams operations on the higher K -groups

$$\Psi_{GS}^k : K_m(X) \rightarrow K_m(X).$$

Gillet and Soulé checked that these maps satisfy the identities of a special lambda ring.

4.2. Vector bundles over a simplicial scheme. Let X be a simplicial scheme, with face maps denoted by d_i and degeneracy maps by s_i . A *vector bundle* E over X consists of a collection of vector bundles $E_n \rightarrow X_n$, $n \geq 0$, together with isomorphisms $d_i^* E_n \cong E_{n+1}$ and $s_i^* E_{n+1} \cong E_n$ for all face and degeneracy maps. Moreover, these isomorphisms should satisfy the simplicial identities. By a *morphism of vector bundles* we mean a collection of morphisms at each level, compatible with these isomorphisms. An *exact sequence* of vector bundles is an exact sequence at every level.

Let $\text{Vect}(X)$ be the exact category of vector bundles over X and consider the algebraic K -groups of $\text{Vect}(X)$, $K_m(\text{Vect}(X))$. These can be computed as the homotopy groups of the simplicial set $S(\text{Vect}(X))$ given by the Waldhausen construction.

For every simplicial scheme X and every $n \geq 0$, there is a natural simplicial map

$$S(\text{Vect}(X)) \rightarrow S(\text{Vect}(X_n)).$$

By the definition of vector bundles over simplicial schemes, it induces a simplicial map

$$S.(\text{Vect}(X.)) \rightarrow \varprojlim_n S.(\text{Vect}(X_n)),$$

which induces a morphism

$$K_m(\text{Vect}(X.)) \xrightarrow{\psi} K_m(X.), \quad m \geq 0.$$

At the zero level, $K_0(\text{Vect}(X.))$ is the Grothendieck group of the category of vector bundles over $X.$, and hence it has a λ -ring structure.

In the particular situation where $X.$ is a simplicial object in $\text{ZAR}(S)$, with S a finite dimensional noetherian scheme, the above morphism can be described as follows (see [GS99]).

Let $N.U$ denote the nerve of a covering U of $X.$ Let $E.^M$ denote the universal vector bundle over $B.GL_{M/S}$ and let $E.$ be a rank N vector bundle over $X.$. Then, there exists a hypercovering $p : N.U \rightarrow X.$ and a classifying map $\chi : N.U \rightarrow B.GL_{M/S}$, for $M \geq N$, such that $p^*(E.) = \chi^*(E.^M)$. The induced map

$$\chi : N.U \rightarrow \{N\} \times \mathbb{Z}_\infty B.GL_M \rightarrow \mathbb{Z} \times \mathbb{Z}_\infty B.GL_M \rightarrow \mathbb{K}.$$

in $\text{ZAR}(S)$, defines an element χ in $H^0(N.U, \mathbb{K}.) = H^0(X., \mathbb{K}.) = K_0(X.)$, which is $\psi([E.])$. This description also shows that the morphism factorizes through the limit

$$\psi : K_0(\text{Vect}(X.)) \rightarrow \varinjlim_{\vec{M}} H^0(X., \mathbb{K}^M) \rightarrow H^0(X., \mathbb{K}).$$

When $X. = B.GL_{N/S}$, we obtain that

$$\begin{aligned} \psi(E.^N - N) &= j_N \in K_0(B.GL_{N/S}), \\ \psi(E.^N - N) &= i_N \in \varinjlim_{\vec{M}} H^0(B.GL_{N/S}, \mathbb{K}^M). \end{aligned}$$

Here N is the trivial bundle of rank N . Clearly, the trivial bundle of rank $r \geq 0$ in $B.GL_{N/S}$, is mapped to u_r .

Consider the λ -ring structure in $K_0(\text{Vect}(B.GL_{N/S}))$ and denote by Ψ^k the corresponding Adams operations. Gillet and Soulé proved in [GS99] section 5 that there are equalities

$$\begin{aligned} \varphi(\Psi^k(id_N - N)) &= \psi(\Psi^k(E.^N - N)), \\ \varphi(\lambda^k(id_N - N)) &= \psi(\lambda^k(E.^N - N)). \end{aligned}$$

Moreover, one can easily check that $\varphi(\Psi^k(id_N - N)) = \Psi_{GS}^k(i_N)$, and $\varphi(\lambda^k(id_N - N)) = \lambda_{GS}^k(i_N)$. Therefore,

$$\Psi_{GS}^k(i_N) = \psi(\Psi^k(E.^N - N)), \quad \text{and} \quad \lambda_{GS}^k(i_N) = \psi(\lambda^k(E.^N - N)).$$

Also, it holds by definition that

$$\Psi_{GS}^k(u_r) = \psi(\Psi^k(u_r)) = u_{\psi^k(r)}, \quad \text{and} \quad \lambda_{GS}^k(u_r) = \psi(\lambda^k(u_r)) = u_{\lambda^k(r)}.$$

4.3. Uniqueness theorems. Let S be a finite dimensional noetherian scheme. Fix \mathbf{C} a Zariski subsite of $\text{ZAR}(S)$ as in section 3.4. The following theorems are a consequence of theorems 3.17 and 3.18 applied to the present situation.

Theorem 4.1 (Lambda operations). *Let $\{\rho_N : \mathbb{K}^N \rightarrow \mathbb{K}.\}_{N \geq 1}$ be a weakly additive system of maps with respect to the H -sum of $\mathbb{K}.$. Let ρ be the induced morphism*

$$\rho : \varinjlim_{\vec{M}} H^*(-, \mathbb{K}^M) \rightarrow H^*(-, \mathbb{F}).$$

If

$$\blacktriangleright \rho(i_N) = \psi(\lambda^k(E.^N - N)), \text{ and,}$$

$$\blacktriangleright \rho(u_r) = u_{\lambda^k(r)},$$

then, ρ agrees with $\lambda_{GS}^k : K_m(X) \rightarrow K_m(X)$, for every degenerate simplicial scheme X in \mathbf{C} .

Theorem 4.2 (Adams operations). *Let $\{\rho_N : \mathbb{K}_N \rightarrow \mathbb{K}\}_{N \geq 1}$ be a weakly additive system of maps with respect to the H -sum of \mathbb{K} . Let ρ be the induced morphism*

$$\lim_{\vec{M}} H^*(-, \mathbb{K}^M) \rightarrow H^*(-, \mathbb{F}).$$

If

$$\blacktriangleright \rho(i_N) = \psi(\Psi^k(E_N - N)), \text{ and,}$$

$$\blacktriangleright \rho(u_r) = u_{\psi^k(r)},$$

then, ρ agrees with $\Psi_{GS}^k : K_m(X) \rightarrow K_m(X)$, for every degenerate simplicial scheme X in \mathbf{C} .

Since the Adams operations are group morphisms, it is natural to expect that they will be induced by H -space maps

$$\mathbb{K} \rightarrow \mathbb{K}.$$

in $\text{Ho}(\mathbf{sT}_*)$. The next two corollaries follow easily from the last theorem.

Corollary 4.3. *Let $\rho : \mathbb{K} \rightarrow \mathbb{K}$ be an H -space map in the homotopy category of simplicial sheaves on \mathbf{C} . If*

$$\blacktriangleright \rho(j_N) = \psi(\Psi^k(E_N - N)), \text{ and,}$$

$$\blacktriangleright \rho(u_r) = \psi(\Psi^k(u_r)),$$

then ρ agrees with the Adams operation Ψ_{GS}^k , for all degenerate simplicial schemes in \mathbf{C} .

Let $S\mathcal{P}$ denote the Waldhausen simplicial sheaf on $ZAR(S)$ given by

$$X \mapsto S\mathcal{P}(X) = S(X).$$

Corollary 4.4. *Let $\rho : S\mathcal{P} \rightarrow S\mathcal{P}$ be an H -space map in $\text{Ho}(\mathbf{sT}_*)$. If for some $k \geq 1$ there is a commutative square*

$$\begin{array}{ccc} K_0(\text{Vect}(B.GL_{N/S})) & \xrightarrow{\psi} & K_0(B.GL_{N/S}) \\ \Psi^k \downarrow & & \downarrow \rho \\ K_0(\text{Vect}(B.GL_{N/S})) & \xrightarrow{\psi} & K_0(B.GL_{N/S}), \end{array}$$

then ρ agrees with the Adams operation Ψ_{GS}^k , for all degenerate simplicial schemes in \mathbf{C} .

Therefore, there is a unique way to extend the Adams operations from the Grothendieck group of simplicial schemes to higher K -theory by means of a map $S\mathcal{P} \rightarrow S\mathcal{P}$ in the homotopy category of simplicial sheaves.

Grayson, in [Gra92], defines the Adams operations for the K -groups of any exact category with a suitable notion of tensor, symmetric and exterior product. The category of vector bundles over a scheme satisfies the required conditions, as well as the category of vector bundles over a simplicial scheme. For every scheme X , he constructs

\blacktriangleright two $(k-1)$ -simplicial sets, $S.\tilde{G}^{(k-1)}(X)$ and $\text{Sub}_k(X)$, whose diagonals are weakly equivalent to $S(X)$, and

\blacktriangleright a $(k-1)$ -simplicial map $\text{Sub}_k(X) \xrightarrow{\Psi^k} S.\tilde{G}^{(k-1)}(X)$.

His construction is functorial on X and hence induces a map of presheaves.

Grayson has already checked that the operations that he defined induce the usual ones for the Grothendieck group of a suitable category \mathcal{P} . Therefore, since the conditions of proposition 4.4 are fulfilled, we obtain the following corollary.

Corollary 4.5. *Let S be a finite dimensional noetherian scheme. The Adams operations defined by Grayson in [Gra92] agree with the Adams operations defined by Gillet and Soulé in [GS99], for every scheme in $\text{ZAR}(S)$. In particular, they satisfy the usual identities for schemes in $\text{ZAR}(S)$.*

Grayson did not prove that his operations satisfied the identities of a lambda ring. It follows from the previous corollary that they are satisfied for finite dimensional noetherian schemes.

5. MORPHISMS BETWEEN K-THEORY AND COHOMOLOGY

5.1. Sheaf cohomology as a generalized cohomology theory. Fix \mathbf{C} to be a subsite of the big Zariski site $\text{ZAR}(S)$, as in section 3.4.

Consider the Dold-Puppe functor $\mathcal{K}(\cdot)$ (see [DP61]), which associates to every cochain complex of abelian groups concentrated in non-positive degrees, G^* , a simplicial abelian group $\mathcal{K}(G)$, pointed by zero. It satisfies the property that $\pi_i(\mathcal{K}(G), 0) = H^{-i}(G^*)$.

Now let G^* be an arbitrary cochain complex. Let $(\tau_{\leq n} G)[n]^*$ be the truncation at degree n of G^* followed by the translation by n . That is,

$$(\tau_{\leq n} G)[n]^i = \begin{cases} G^{i+n} & \text{if } i < 0, \\ \ker(d : G^n \rightarrow G^{n+1}) & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

One defines a simplicial abelian group by

$$\mathcal{K}(G)_n := \mathcal{K}((\tau_{\leq n} G)[n]).$$

The simplicial abelian groups $\mathcal{K}(G)_n$ form an infinite loop spectrum. Moreover, this construction is functorial on G .

Let \mathcal{F}^* be a cochain complex of sheaves of abelian groups in \mathbf{C} , and let $\mathcal{K}(\mathcal{F})_*$ be the infinite loop spectrum obtained applying section-wise the construction above. For every n , $\mathcal{K}(\mathcal{F})_n$ is an H -space, since it is a simplicial sheaf of abelian groups.

Lemma 5.1 ([HW98] Prop. B.3.2). *Let \mathcal{F}^* be a bounded below complex of sheaves on \mathbf{C} and let X be a scheme in the underlying category. Then, for all $m \in \mathbb{Z}$,*

$$H^m(X, \mathcal{K}(\mathcal{F})_*) \cong H_{\text{ZAR}}^m(X, \mathcal{F}^*).$$

Here, the right hand side is the usual Zariski cohomology and the left hand side is the generalized cohomology of the simplicial sheaf of groups $\mathcal{K}(\mathcal{F})$. Observe that since $\mathcal{K}(\mathcal{F})$ is an infinite loop space, we can consider generalized cohomology groups for all integer degrees. Thus we see that the usual Zariski cohomology can be expressed in terms of generalized sheaf cohomology using the Dold-Puppe functor.

5.2. Uniqueness of characteristic classes. Now fix a bounded below graded complex of sheaves $\mathcal{F}^*(*)$ of abelian groups, giving a twisted duality cohomology theory in the sense of Gillet, [Gil81]. In loc. cit., Gillet constructed *Chern classes* for higher K -theory. They are given by a map of spaces

$$(5.2) \quad c_j : \mathbb{K} \rightarrow \mathcal{K}(\mathcal{F}(j)[2j]), \quad j \geq 0.$$

More specifically, they are given by a map

$$\mathbb{Z}_{\infty} B.GL \rightarrow \mathcal{K}(\mathcal{F}(j)[2j])$$

extended trivially over the \mathbb{Z} component of \mathbb{K} . By example 3.6, these maps are weakly additive. In fact, they are weakly additive also with respect to the H -sum of $\mathcal{K}(\mathcal{F}(j)[2j])$ (see remark 3.8).

For any space X , the map induced after taking generalized cohomology,

$$c_j : K_m(X) \rightarrow H_{\text{ZAR}}^{2j-m}(X, \mathcal{F}^*(j)), \quad j \geq 0,$$

is called the j -th *Chern class*. They are group morphisms for $m > 0$ but only maps for $m = 0$. In this last case, for any vector bundle E over a scheme X , $c_j(E)$ is the standard j -th Chern class taking values in the given cohomology theory.

Using the standard formulas on the Chern classes, one obtains the *Chern character*

$$\text{ch} : K_m(X) \rightarrow \prod_{j \geq 0} H_{\text{ZAR}}^{2j-m}(X, \mathcal{F}^*(j)) \otimes \mathbb{Q},$$

which is now a group morphism for all $m \geq 0$. It is induced by an H -space map

$$\text{ch} : \mathbb{K} \rightarrow \prod_{j \geq 0} \mathcal{K}(\mathcal{F}(j)[2j]) \otimes \mathbb{Q}.$$

The restriction of ch to $K_0(X)$ is the usual Chern character of a vector bundle.

We will now state the theorems equivalent to theorems 4.1 and 4.2, for maps from K -theory to cohomology. In order to do this, we should first understand better $c_j(i_N)$ and $\text{ch}(i_N)$ for all j, N . This will be achieved by means of the Grassmanian schemes.

Denote by

$$\begin{aligned} \mathcal{F}^*(*) &= \prod_{i \geq 0, j \in \mathbb{Z}} \mathcal{F}^i(j), \\ H_{\text{ZAR}}^*(X, \mathcal{F}^*(*)) &= \prod_{i \geq 0, j \in \mathbb{Z}} H_{\text{ZAR}}^i(X, \mathcal{F}^*(j)). \end{aligned}$$

Let $Gr(N, k) = Gr_{\mathbb{Z}}(N, k) \times_{\mathbb{Z}} S$ be the Grassmanian scheme over S . This is a projective scheme over S . Consider $E_{N,k}$ the rank N universal bundle of $Gr(N, k)$ and $\{U_k\} \xrightarrow{p} Gr(N, k)$ its standard trivialization. There is a classifying map of the vector bundle $E_{N,k}$, $\varphi_k : N.U_k \rightarrow B.GL_{N/S}$, satisfying $p^*(E_{N,k}) = \varphi_k^*(E^N)$. This map induces a map in the Zariski cohomology

$$H_{\text{ZAR}}^*(B.GL_{N/S}, \mathcal{F}^*(*)) \xrightarrow{\varphi_k^*} H_{\text{ZAR}}^*(N.U_k, \mathcal{F}^*(*)) \cong H_{\text{ZAR}}^*(Gr(N, k), \mathcal{F}^*(*)).$$

Moreover, for each m_0 , there exists k_0 such that if $m \leq m_0$ and $k \geq k_0$, φ_k^* is an isomorphism on the m -th cohomology group.

Proposition 5.3. *Let $\chi_1 = \{\chi_1^N\}$ and $\chi_2 = \{\chi_2^N\}$ be two weakly additive systems of maps*

$$\chi_i^N : \mathbb{K}^N \rightarrow \mathcal{K}(\mathcal{F}^*(i)), \quad i = 1, 2,$$

with respect to the same operation. Then, the induced maps

$$\chi_1, \chi_2 : K_m(X) \rightarrow H_{\text{ZAR}}^*(X, \mathcal{F}^*(*))$$

agree for every scheme X in \mathbf{C} , if and only if they agree for $X = Gr(N, k)$, for all N and k .

Proof. One implication is obvious. For the other implication, fix m_0 and let k_0 be an integer such that for every $k \geq k_0$ there is an isomorphism at the m_0 level. Then, there are commutative diagrams

$$\begin{array}{ccc} \lim_{\overrightarrow{M}} H^{-m_0}(B.GL_{N/S}, \mathbb{K}^M) & \xrightarrow{\chi_1, \chi_2} & H_{\text{ZAR}}^*(B.GL_{N/S}, \mathcal{F}^*(*)) \\ \varphi_k^* \downarrow & & \cong \downarrow \varphi_k^* \\ \lim_{\overrightarrow{M}} H^{-m_0}(N.U_k, \mathbb{K}^M) & \xrightarrow{\chi_1, \chi_2} & H_{\text{ZAR}}^{-m_0}(N.U_k, \mathcal{F}^*(*)) \\ p^* \uparrow \cong & & \cong \uparrow p^* \\ H^{-m_0}(Gr(N, k), \mathbb{K}) & \xrightarrow{\chi_1, \chi_2} & H_{\text{ZAR}}^{-m_0}(Gr(N, k), \mathcal{F}^*(*)). \end{array}$$

By theorem 3.16, $\chi_1 = \chi_2$ for all schemes X , if they agree for $B.GL_{N/S}$ for all $N \geq 1$. Let $x \in \lim_{\overrightarrow{M}} H^{-m_0}(B.GL_{N/S}, \mathbb{K}^M)$. Then,

$$\begin{aligned} \chi_1(x) = \chi_2(x) &\Leftrightarrow (p^*)^{-1} \varphi_k^* \chi_1(x) = (p^*)^{-1} \varphi_k^* \chi_2(x) \\ &\Leftrightarrow \chi_1(p^*)^{-1} \varphi_k^*(x) = \chi_2(p^*)^{-1} \varphi_k^*(x), \end{aligned}$$

and since they agree for the Grassmanians, the proposition is proved. \square

The following two theorems follow from the results 3.16 and 3.17, together with the preceding proposition.

Theorem 5.4 (Chern classes). *There is a unique way to extend the j -th Chern class of vector bundles over schemes in \mathbf{C} , by means of a weakly additive system of maps $\{\rho_N : \mathbb{K}^N \rightarrow \mathcal{K}(\mathcal{F}(j)[2j])\}_{N \geq 1}$ with respect to the H -space operation in $\mathcal{K}(\mathcal{F}(*))$.*

Observe that it follows from the theorem that any weakly additive collection of maps with respect to the H -space operation of $\mathcal{K}(\mathcal{F}(*))$, inducing the j -th Chern class, is necessarily trivial on the \mathbb{Z} -component for $j > 0$.

Theorem 5.5 (Chern character). *Let*

$$\mathbb{K} \longrightarrow \prod_{j \in \mathbb{Z}} \mathcal{K}(\mathcal{F}(j)[2j])$$

be an H -space map in $\text{Ho}(\mathbf{sT}_)$. The induced morphisms*

$$K_m(X) \rightarrow \prod_{j \in \mathbb{Z}} H_{\text{ZAR}}^{2j-m}(X, \mathcal{F}^*(j))$$

agree with the Chern character defined by Gillet in [Gil81] for every scheme X , if and only if, the induced map

$$K_0(X) \rightarrow \prod_{j \in \mathbb{Z}} H_{\text{ZAR}}^{2j}(X, \mathcal{F}^*(j))$$

is the Chern character for $X = \text{Gr}(N, k)$, for all N, k .

Corollary 5.6. *There is a unique way to extend the standard Chern character of vector bundles over schemes in \mathbf{C} , by means of an H -space map*

$$\rho : \mathbb{K} \rightarrow \prod_{j \in \mathbb{Z}} \mathcal{K}(\mathcal{F}(j)[2j]).$$

We deduce from these theorems that any simplicial sheaf map

$$S.\mathcal{P} \rightarrow \mathcal{K}(\mathcal{F}(*))$$

that induces either the Chern character or any Chern class map at the level of $K_0(X)$, induces the Chern character or the Chern class map on the higher K -groups of X .

Remark 5.7. Let \mathbf{C} be the site of smooth complex varieties and let $\mathcal{D}^*(*)$ be a graded complex computing absolute Hodge cohomology. Burgos and Wang, in [BW98], constructed a simplicial sheaf map $S.\mathcal{P} \rightarrow \mathcal{K}(\mathcal{D}^*(*))$ which induces the Chern character on any smooth proper complex variety. A consequence of the last corollary is that their definition agrees with the *Beilinson regulator* (the Chern character for absolute Hodge cohomology).

This is not a new result. Using other methods, Burgos and Wang already proved that the morphism they defined was the same as the Beilinson regulator. The result is proved there by means of the bisimplicial scheme $B.P$ introduced by Schechtman in [Sch87]. This introduced an unnecessary delooping, making the proof generalizable only to sheaf maps inducing group morphisms and introducing irrelevant ingredients to the proof.

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